

Advances on Matroid Secretary Problems: Free Order Model and Laminar Case

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Abstract

The most well-known conjecture in the context of matroid secretary problems claims the existence of a constant-factor approximation applicable to any matroid. Whereas this conjecture remains open, modified forms of it were shown to be true, when assuming that the assignment of weights to the secretaries is not adversarial but uniformly random [19, 17]. However, so far, there was no variant of the matroid secretary problem with adversarial weight assignment for which a constant-factor approximation was found. We address this point by presenting a 9-approximation for the *free order model*, a model suggested shortly after the introduction of the matroid secretary problem, and for which no constant-factor approximation was known so far. The free order model is a relaxed version of the original matroid secretary problem, with the only difference that one can choose the order in which secretaries are interviewed.

Furthermore, we consider the classical matroid secretary problem for the special case of laminar matroids. Only recently, a constant-factor approximation has been found for this case, using a clever but rather involved method and analysis [12] that leads to a $16000/3$ -approximation. This is arguably the most involved special case of the matroid secretary problem for which a constant-factor approximation is known. We present a considerably simpler and stronger $3\sqrt{3}e \approx 14.12$ -approximation, based on reducing the problem to a matroid secretary problem on a partition matroid.

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1 Introduction

The secretary problem is a classical online selection problem of unclear origin [6, 8, 9, 10, 16]. In its original form, the task is to choose the best out of n secretaries, also called *elements* or *items*. Secretaries arrive (or are interviewed) one by one in random order. As soon as a secretary arrives, it can be ranked against all previously seen secretaries. Then, before the next one arrives, one has to decide irrevocably whether to choose the current secretary or not. There is a classical algorithm, that selects the best secretary with probability $1/e$ [6], and this is known to be asymptotically optimal. In its initial form, the secretary problem was essentially a stopping time problem, and not surprisingly, it mainly attracted the interest of probabilists.

Recently, secretary problems enjoyed a revival, and various generalized types of secretary problems were studied. These developments are strongly motivated by a close connection to online mechanism design, where a good is sold to agents arriving online [13, 2]. Here, the agents correspond to the secretaries and they reveal prices that they are willing to pay in exchange for goods. This leads to secretary problems where more than one secretary can be chosen. The most canonical generalization asks to hire k out of n secretaries, each revealing a non-negative weight upon arrival, and the goal is to hire a maximum weight subset of k secretaries. This interesting variant was introduced and studied by Kleinberg [13], who presented a $(1 - O(1/\sqrt{k}))$ -approximation for this setting. However, in many applications, additional constraints have to be imposed on the elements that can be chosen. A very general class of constrained secretary problems, where the chosen elements have to form an independent set of a given matroid $M = (N, \mathcal{I})$, was introduced by Babaioff, Immorlica and Kleinberg [2]¹. This setting, now generally termed *matroid secretary problem*, covers at the same time many interesting cases and has a rich structure that can be exploited to design strong approximation algorithms.

To give a concrete example of a matroid secretary problem, and to motivate some of our results, consider the following connection problem. Given is an undirected graph $G = (V, E)$, representing a communication network, with non-negative edge-capacities $c : E \rightarrow \mathbb{N}$ and a server $r \in V$. Clients, which are the equivalent of candidates in the secretary problem, reside at vertices of the graph and are interested to connect to the server r via a unit-capacity path. The number of clients and their locations are known. Each client has a price that she is willing to pay to connect to the server. These prices are unknown and no assumptions are made on them except for being non-negative. Clients then reveal themselves one by one in random order, announcing their price. As in the secretary problem, whenever a client reveals herself, the network operator has to decide irrevocably before the next client appears whether to serve this client and receive the announced price. The goal is to choose a maximum weight subset of clients that can be served simultaneously without exceeding the given capacities c . It is well-known that the constraint imposed by the limited capacity on the clients that can be chosen is a special type of matroid constraint, namely a gammoid constraint [18].

Many special cases and variants of the matroid secretary problem have been studied recently. For the classical matroid secretary problem, as discussed above, the currently best approximation algorithm was recently introduced by Chakraborty and Lachish [4], leading to an $O(\sqrt{\log(\rho)})$ -approximation, where ρ is the rank of the matroid². This improved on an earlier $O(\log(\rho))$ -approximation of Babaioff, Immorlica and Kleinberg [2]. Babaioff et al. conjectured that there is an $O(1)$ -approximation for the matroid secretary problem. This conjecture remains open and is arguably the currently most important open question regarding the matroid secretary problem.

¹A matroid $M = (N, \mathcal{I})$ consists of a finite set N , called the *ground set*, and a non-empty family $\mathcal{I} \subseteq 2^N$ of subsets of N , called *independent sets*, satisfying: (i) $I \in \mathcal{I}, J \subseteq I \Rightarrow J \in \mathcal{I}$, and (ii) $I, J \in \mathcal{I}, |I| > |J| \Rightarrow \exists f \in I \setminus J$ with $J \cup \{f\} \in \mathcal{I}$.

²The rank of a matroid is the cardinality of a basis, i.e., a maximal independent set. More generally, the rank function $r : 2^N \rightarrow \mathbb{Z}_{\geq 0}$ of a matroid $M = (N, \mathcal{I})$ returns for each set $S \subseteq N$ the size of a maximal independent set contained in S , i.e., $r(S) = \max\{|I| \mid I \in \mathcal{I}, I \subseteq S\}$. Hence, $\rho = r(N)$.

Motivated by this conjecture, many interesting advances have been made to obtain constant-factor approximations, either for special cases of the matroid secretary problem or variants thereof. In particular, constant-factor approximations have been found for graphic matroids [2, 15] (currently best approximation factor: $2e$), transversal matroids [2, 5, 15] (8-approximation), co-graphic matroids [19] ($3e$ -approximation), linear matroids with at most k non-zero entries per column [19] (ke -approximation), and most recently laminar matroids [12] ($16000/3$ -approximation). For most of the above special cases, strong approximation algorithms have been found, typically based on very elegant techniques. However for the laminar matroid, only a considerably higher approximation factor is known due to Im and Wang [12], using a very clever but quite involved method and analysis.

Furthermore, variants of the matroid secretary problem have been investigated that assume random instead of adversarial assignment of the weights, and for which $O(1)$ -approximations can be obtained without any restriction on the underlying matroid. Recall that the classical matroid secretary problem does not assume anything about how weights are assigned to the elements, which means that we have to assume a worst-case, i.e., *adversarial*, weight assignment. However, the order in which the elements reveal themselves is assumed to be random. Soto [19] considered the variant where not only the arrival order of the elements is assumed to be uniformly random but also the assignment of the weights to the elements, and presented a $2e^2/(e-1)$ -approximation for this case. More precisely, in this model, the weights can still be chosen by an adversary, but these weights are then assigned to the elements of the matroid uniformly at random. Building on Soto's work, Vondrák and Oveis Gharan [17] showed that a $40e/(e-1)$ -approximation can even be obtained when the arrival order of the elements is adversarial, and the assignment of the weights is still assumed to be uniformly at random. Hence, this model is somehow the opposite of the classical matroid secretary problem, where assignment is adversarial and arrival order is random.

However, so far, no progress has been made in variants where the assignment remains adversarial. One such variant, which was suggested shortly after the introduction of the matroid secretary problem [14], assumes that the order in which elements appear can be chosen by the algorithm. More precisely, in this model, which we call the *free order model*, whenever an element has to reveal itself, the algorithm can choose the element to be revealed. E.g. in the above network connection problem, one could decide at each step which is the next client to reveal its price, by using for this decision the network structure and the elements observed so far. A main further complication when dealing with adversarial assignments—as in the free order model—contrary to random assignment, is that the knowledge of the initial structure of the matroid seems to be of little help. This is due to the fact that an adversary can assign a weight of zero to most elements of the matroid, and only give a non-negative weight to a selected subset $A \subseteq N$ of elements. Hence, the problem essentially reduces to the restriction $M|_A$ of the matroid M over the elements A . However, the structure of $M|_A$ is essentially impossible to guess from M . This is in stark contrast to models with random assignment, e.g., in the model considered by Soto, he was able to design an algorithm that right at the start exploits the given structure of the matroid M , by partitioning N and solving a standard single secretary problem on each part of the partition. Hence, different approaches are needed for adversarial weight assignments.

We are interested in the following two questions. First, is there an $O(1)$ -approximation for the free order model? Second, we are interested in getting a better understanding of the laminar case of the classical secretary problem, with the goal to find considerably stronger and simpler procedures.

As it is common in this context, when we talk about a c -approximation we always compare against the *offline* optimum solution, i.e., the maximum weight independent set. In this type of analysis, known as *competitive analysis*, a c -approximation is also called a *c-competitive algorithm*.

Our results and techniques

We present a 9-approximation for the free order model, thus obtaining the first $O(1)$ -approximation for a variant of the matroid secretary problem with adversarial weight assignment, without any restriction on the underlying matroid. We remark that this algorithm can in particular deal with the mentioned network connection problem, when the order, in which the network operator negotiates with the clients, can be chosen. Previously, no matroid secretary model with adversarial weight assignment was known to admit an $O(1)$ -approximation for this problem setting.

Our algorithm follows a quite intuitive idea, which, interestingly, does not work in the traditional matroid secretary problem. More precisely, in a first phase, we draw each element with probability 0.5 to obtain a set $A \subseteq N$, without selecting any element of A . Let OPT_A be the best offline solution in A . We call an element $f \in N \setminus A$ *good*, if it can be used to improve OPT_A , in the sense that either $\text{OPT}_A \cup \{f\}$ is independent or there is an element $g \in \text{OPT}_A$ such that $(\text{OPT}_A \setminus \{g\}) \cup \{f\}$ is independent and has a higher value than OPT_A . In the second phase we go through the remaining elements $N \setminus A$, drawing element by element in a well-chosen way to be specified soon. We accept an element $f \in N \setminus A$ if it is good and does not destroy independence when added to the elements accepted so far. Our approach fails if elements are drawn randomly in the second phase. The main problem when drawing randomly, is that we may accept good elements of relatively low value that may later *block* some high-valued good elements, in the sense that they cannot be added anymore without destroying independence of the selected elements. To overcome this problem, we determine after the first phase a specific order according to which elements will be drawn in the second phase. The idea is to first draw elements of $N \setminus A$ that are in the span of elements of A of high weight. More precisely, let $A = \{a_1, \dots, a_m\}$ be the numbering of the elements of A according to decreasing weights. In the second phase we start by drawing elements of $(N \setminus A) \cap \text{span}(\{a_1\})$, then $(N \setminus A) \cap \text{span}(\{a_1, a_2\})$, and so on³. The intuition is that if there is a set $S \subseteq N$ with a high density of high-valued elements, then it is likely that many elements of S are part of A . Hence, the high-valued elements of A span further high-valued elements in S . Thus, by the above order, we are likely to draw high-valued elements of S early, before they can be blocked by the inclusion of lower-valued elements.

Similar to previous secretary algorithms, we show that our algorithm is a constant-factor approximation by proving that each element $f \in \text{OPT}$ of the global offline optimum OPT will be chosen with probability at least $1/9$. However, the way we prove this is based on a novel approach. Broadly speaking, we show that for any element $f \in \text{OPT}$ there is a threshold weight \bar{w}_f such that with constant positive probability we have simultaneously: (i) $f \notin A$, (ii) f is spanned by the elements in A with weight $\geq \bar{w}_f$, and (iii) good elements considered in the second phase with weight at least \bar{w}_f do not block f . From this we can observe that f gets selected with constant probability. An interesting aspect of this analysis is that several probabilities of interest that appear in our analysis are very hard to compute exactly. E.g., even when all weights are known and a threshold \bar{w}_f is given, it is in general $\#P$ -hard to compute the probability that f is in the span of all elements of A of weight at least \bar{w}_f ⁴. Still, we can show that a good threshold weight \bar{w}_f exists, which is all we need to guarantee that our algorithm is a constant-factor approximation.

Furthermore, we present a new approach to deal with laminar matroids in the classical matroid secretary model. Our technique leads to a $3\sqrt{3}e \approx 14.12$ -approximation, thus considerably improving on the $16000/3 \approx 5333$ -approximation of Im and Wang [12]. Our main contribution here is to present a simple way to transform the matroid secretary problem on a laminar matroid M to a matroid secretary problem on

³We recall that $\text{span}(S)$ for $S \subseteq N$ is the unique maximal set $U \supseteq S$ having the same rank as S .

⁴Consider for example the graphic matroid with underlying graph $G = (V, E)$. Here, the question whether some edge $\{s, t\} \in E$ is in the span of a random set of edge $A \subseteq E$ containing each edge with probability 0.5, reduces to the question of whether A contains an s - t path. This is the well-known $\#P$ -hard s - t reliability problem [20].

a unitary partition matroid⁵ M_P losing only a small constant factor of $3\sqrt{3} \approx 5.2$. As it is well-known, the secretary problem on M_P can then simply be solved by applying the classical e -approximation for the standard secretary problem to each partition of M_P . We first observe a constant fraction of all elements, on the basis of which a partition matroid M_P on the remaining elements is then constructed. To assure feasibility, M_P is defined such that each independent set of M_P is as well an independent set of M . To best convey the main ideas of our procedure, we first present a very simple method to obtain a weaker $27e/2 \approx 36.7$ -approximation, which already improves considerably on the $16000/3$ -approximation of Im and Wang. We then show how to strengthen this approach by using a stronger partition matroid M_P and by applying a sharper analysis, leading to the claimed $3\sqrt{3}e$ -approximation.

We remark that the algorithms we present do not need to observe the exact weights of the items when they reveal themselves, but only need to be able to *compare* the weights of elements observed so far. This is a common feature of many algorithms developed in the context of the matroid secretary problem.

To simplify the exposition, we assume that all weights are distinct, i.e., they induce a linear order on the elements. This implies in particular, that there is a unique maximum weight independent set. The general case with possibly equal weights easily reduces to this case by breaking ties arbitrarily between elements of equal weight to obtain a linear order.

Related work

We briefly mention some further related work. Recently, the matroid secretary problem was also considered with submodular objective functions instead of linear ones. For this setting, constant-factor approximations have been found for knapsack constraints, uniform matroids, and more generally for partition matroids if the submodular objective function is furthermore monotone [3, 7, 11].

Additionally, variations of the matroid secretary problem have been considered with restricted knowledge on the underlying matroid type. This includes the case where no prior knowledge of the underlying matroid is assumed except for the size of the ground set. Or even more extremely, the case without even knowing the size of the ground set. For more information on such variations we refer to the excellent overview in [17].

Organization of the paper

Our constant-factor approximation for the free order case is presented in Section 2. Section 3 discusses the classical matroid secretary problem restricted to laminar matroids. We refer the reader to [18] for further information on matroids.

2 A 9-approximation for the free order model

To simplify the writing we use “+” and “−” for the addition and subtraction of single elements from a set, i.e., $S + f - g = (S \cup \{f\}) \setminus \{g\}$. Algorithm 1 describes our 9-approximation for the free order model. It operates in two phases.

As mentioned previously, a *good* element $f \in N \setminus A$ is an element that allows for improving the maximum weight independent set in A . Using standard results on matroids, an element f is good if either $f \notin \text{span}(A)$, or if there is an index $i \in \{1, \dots, m\}$ such that $f \in \text{span}(A_i) \setminus \text{span}(A_{i-1})$ and $w(f) > w(a_i)$. Hence, our algorithm indeed only accepts good elements.

⁵A *unitary partition matroid* is a partition matroid where up to one element can be chosen in each set of the partition.

Algorithm 1 A 9-approximation for the free order model.

1. **Draw** each element with probability 0.5 to obtain $A \subseteq N$, without selecting any element of A . We number the elements of $A = \{a_1, \dots, a_m\}$ in decreasing order of weights. Define $A_i = \{a_1, \dots, a_i\}$, with $A_0 = \emptyset$.
Initialize: $I \leftarrow \emptyset$.
 2. **For** $i = 1$ to m :
 draw one by one (in any order) all elements $f \in (\text{span}(A_i) \setminus \text{span}(A_{i-1})) \setminus A$.
 if $I + f \in \mathcal{I}$ and $w(f) > w(a_i)$, **then** $I = I + f$.
 For all remaining elements $f \in N \setminus \text{span}(A)$ (drawn in any order):
 if $I + f \in \mathcal{I}$, **then** $I = I + f$.
 Return I
-

To show that Algorithm 1 is a 9-approximation, we show that each element f of the offline optimum OPT will be contained in the set I returned by the algorithm with probability at least $1/9$. We distinguish two cases:

- (i) $\Pr[f \in \text{span}(A - f)] \leq 1/3$, and
- (ii) $\Pr[f \in \text{span}(A - f)] > 1/3$.

The following lemma handles the first case, which is simpler and allows us to highlight some ideas that we will also employ to prove the more interesting second case. Notice that in the following statement we do not even have to assume $f \in \text{OPT}$.

Lemma 1. *Let $f \in N$ with $\Pr[f \in \text{span}(A - f)] \leq 1/3$. Then f is selected by Algorithm 1 with probability at least $1/6$.*

Proof. We start by observing that f is selected by Algorithm 1 if the following three events E_1, E_2 and E_3 happen simultaneously:

- E_1 : $f \notin A$,
- E_2 : $f \notin \text{span}(A - f)$, and
- E_3 : $f \notin \text{span}((N \setminus A) - f)$.

Indeed, if $E_1 \cap E_2$ occurs, then f will be considered during the second for-loop of the second phase of Algorithm 1. Furthermore, adding f at that moment will not violate independence since the elements selected so far are a subset of $N \setminus A$, and if E_3 holds we have $f \notin \text{span}((N \setminus A) - f)$. It therefore suffices to show that the probability of E_1, E_2, E_3 happening simultaneously is at least $1/6$.

Notice that E_1 is independent of E_2, E_3 . Hence,

$$\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \cdot \Pr[E_2 \cap E_3] = \frac{1}{2} \cdot \Pr[E_2 \cap E_3]. \quad (1)$$

Furthermore, we observe that A and $N \setminus A$ have the same distribution since they contain each element of N with probability 0.5. Hence,

$$\Pr[E_3] = \Pr[E_2] = 1 - \Pr[f \in \text{span}(A - f)] \geq \frac{2}{3}.$$

Denoting by $\overline{E_2}$ and $\overline{E_3}$ the complements of E_2 and E_3 , respectively, we thus obtain by the union bound:

$$\begin{aligned}\Pr[E_2 \cap E_3] &= 1 - \Pr[\overline{E_2} \cup \overline{E_3}] \geq 1 - \Pr[\overline{E_2}] - \Pr[\overline{E_3}] \\ &= \Pr[E_2] + \Pr[E_3] - 1 \geq \frac{1}{3}.\end{aligned}$$

Combining the above inequality with (1) we obtain as desired $\Pr[E_1 \cap E_2 \cap E_3] \geq 1/6$. \square

We now consider the case $f \in \text{OPT}$ with $\Pr[f \in \text{span}(A - f)] > 1/3$. Let $N = \{f_1, \dots, f_n\}$ be the numbering of all elements in decreasing order of weights, and let $N_j = \{f_1, \dots, f_j\}$ with $N_0 = \emptyset$. This time, we want to show that with constant probability, f is chosen in the first for-loop of the second phase of Algorithm 1. More precisely, we want to find a good threshold weight \bar{w}_f as discussed in the introduction. For this we determine an index $\bar{j} \in \{1, \dots, n\}$ —and \bar{w}_f will then correspond to $w(f_{\bar{j}})$ —satisfying two properties. First, we want that with constant positive probability, $f \in \text{span}((A \cap N_{\bar{j}}) - f)$. The benefit of having $f \in \text{span}((A \cap N_{\bar{j}}) - f)$ is that if additionally $f \notin A$, then we know that f will be considered in the first for-loop of phase two at some iteration i with $w(a_i) \geq w(f_{\bar{j}})$. Hence, up to that point, only elements with weight at least $w(f_{\bar{j}})$ have been selected. Thus, when having to check whether f can be added without violating independence, only those elements have to be considered. Second, we want that $\Pr[f \in \text{span}((A \cap N_{\bar{j}}) - f)]$ is also bounded away from 1, because this implies that $\Pr[f \notin \text{span}((N_{\bar{j}} \setminus A) - f)] = \Pr[f \notin \text{span}((A \cap N_{\bar{j}}) - f)]$ is some constant > 0 . Whenever $f \notin \text{span}((N_{\bar{j}} \setminus A) - f)$ occurs, then f will not violate independence when added to any set of selected elements with weight at least $w(f_{\bar{j}})$, since they are a subset of $N_{\bar{j}} \setminus A$. Hence, intuitively, for our analysis we want to find an index \bar{j} such that $\Pr[f \in \text{span}((A \cap N_{\bar{j}}) - f)]$ is bounded away from zero and from one. The following lemma shows that such an index indeed exists. In Lemma 3, we then show how the above sketch of our proof can be formalized, and in particular, how to deal with dependencies of the different events discussed above.

For brevity, let $p_j = \Pr[f \in \text{span}((A \cap N_j) - f)]$.

Lemma 2. *Let $f \in N$ with $\Pr[f \in \text{span}(A - f)] \geq 1/3$. There exists an index $\bar{j} \in \{1, \dots, n\}$, such that $p_{\bar{j}} \in [1/3, 2/3]$.*

Proof. By assumption we have

$$p_n = \Pr[f \in \text{span}(A - f)] > 1/3.$$

Furthermore, $p_0 = 0$. Since p_j is increasing in j , to prove the proposition it suffices to show that $p_{j+1} \leq 2/3$, for all $j \in \{0, \dots, n-1\}$ such that $p_j < 1/3$. This indeed holds due to the following:

$$\begin{aligned}p_{j+1} &= \underbrace{\Pr[f_{j+1} \notin A]}_{=0.5} \cdot \underbrace{\Pr[f \in \text{span}((A \cap N_{j+1}) - f) \mid f_{j+1} \notin A]}_{=p_j} + \\ &\quad \underbrace{\Pr[f_{j+1} \in A]}_{=0.5} \cdot \underbrace{\Pr[f \in \text{span}((A \cap N_{j+1}) - f) \mid f_{j+1} \in A]}_{\leq 1} \\ &\leq \frac{1}{2}(p_j + 1).\end{aligned}$$

\square

The following lemma completes the proof of the case $\Pr[f \in \text{span}(A - f)] > 1/3$.

Lemma 3. *Let $f \in \text{OPT}$ with $\Pr[f \in \text{span}(A - f)] > 1/3$. Then f is selected by Algorithm 1 with probability at least $1/9$.*

Proof. Let $\bar{j} \in \{1, \dots, n\}$ be an index with $p_{\bar{j}} \in [1/3, 2/3]$ as claimed by Lemma 2. We start by reasoning that f will be selected by Algorithm 1 if the following three events E_1, E_2 and E_3 happen simultaneously:

- E_1 : $f \notin A$,
- E_2 : $f \in \text{span}((A \cap N_{\bar{j}}) - f)$, and
- E_3 : $f \notin \text{span}((N_{\bar{j}} \setminus A) - f)$.

Notice that $E_1 \cap E_2$ implies that f will be considered during the first for-loop of the second phase of Algorithm 1, at some iteration i with $w(a_i) \geq w(f_{\bar{j}})$. Since the elements selected so far—at the time f is considered—must all have a weight of at least $w(a_i) \geq w(f_{\bar{j}})$, the occurrence of E_3 guarantees that f can be added without violating independence since the selected elements at that point are a subset of $N_{\bar{j}} \setminus A$. Also notice that since $f \in \text{OPT}$ and $f \in \text{span}(A_i)$, we have $w(f) > w(a_i)$, i.e., f is good. Therefore, f indeed gets selected if E_1, E_2, E_3 occur simultaneously. Hence it suffices to show that $E_1 \cap E_2 \cap E_3$ occurs with probability $\geq 1/9$. Again, E_1 is independent of E_2, E_3 , and hence

$$\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \cdot \Pr[E_2 \cap E_3] = \frac{1}{2} \cdot \Pr[E_2 \cap E_3]. \quad (2)$$

To deal with the dependence between the events E_2 and E_3 we invoke the FKG inequality (see [1]). Notice that both events E_2 and E_3 are *increasing* in A , i.e., for any two sets $Q, P \subseteq N$ with $Q \subseteq P$, if E_2 (or E_3) occurs for $A = Q$ then it also occurs if $A = P$. The FKG inequality then implies

$$\Pr[E_2 \cap E_3] \geq \Pr[E_2] \cdot \Pr[E_3]. \quad (3)$$

Furthermore, since $A \cap N_{\bar{j}}$ has the same distribution as $N_{\bar{j}} \setminus A$, we have $\Pr[E_3] = 1 - \Pr[E_2]$. Hence, together with (2) and (3) we obtain

$$\Pr[E_1 \cap E_2 \cap E_3] = \frac{1}{2} \cdot \Pr[E_2] \cdot (1 - \Pr[E_2]).$$

Due to our choice of \bar{j} , we have $\Pr[E_2] \in [1/3, 2/3]$, and hence $\Pr[E_2] \cdot (1 - \Pr[E_2]) \geq 2/9$, thus leading to $\Pr[E_1 \cap E_2 \cap E_3] \geq 1/9$ as desired. \square

Finally combining Lemma 1 and Lemma 3 we obtain the claimed approximation guarantee.

Corollary 4. *Algorithm 1 is a 9-approximation for the free order model.*

3 Classical matroid secretary problem for laminar matroids

Let $M = (N, \mathcal{I})$ be a laminar matroid whose constraints are defined by the laminar family $\mathcal{L} \subseteq 2^N$ with upper bounds b_L for $L \in \mathcal{L}$ on the number of elements that can be chosen from L , i.e., $\mathcal{I} = \{I \subseteq N \mid |I \cap L| \leq b_L \forall L \in \mathcal{L}\}$. Without loss of generality we assume $b_L \geq 1$ for $L \in \mathcal{L}$, since otherwise we can simply remove all elements of L from M . Furthermore, we assume $N \in \mathcal{L}$, since otherwise a redundant constraint $|I \cap N| \leq b_N$ can be added by choosing a sufficiently large right-hand side b_N .

To reduce the matroid secretary problem on M to a problem on a partition matroid, we first number the elements $N = \{f_1, \dots, f_n\}$ such that for any set $L \in \mathcal{L}$, the elements in L are numbered consecutively, i.e., $L = \{f_p, \dots, f_q\}$ for some $1 \leq p < q \leq n$. Figure 1 shows an example of such a numbering.

For the sake of exposition, we start by presenting a conceptually simple algorithm and analysis, based on the introduced numbering of the ground set, that leads to a $27e/2$ -approximation. We later present strengthenings of both the approach and the analysis to obtain the claimed $3\sqrt{3}e$ -approximation. Algorithm 2 describes our $27e/2$ -approximation.

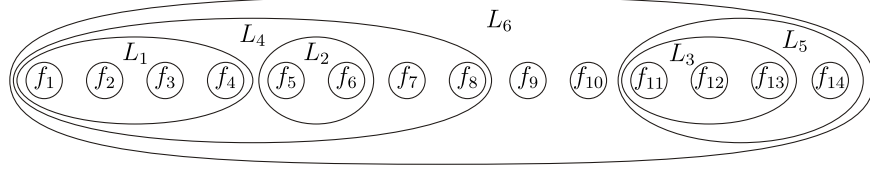


Figure 1: An example of a numbering of the elements of the ground set such that each set $L \in \mathcal{L} = \{L_1, \dots, L_6\}$ of the laminar family contains consecutively numbered elements.

Algorithm 2 A $27e/2$ -approximation for laminar matroids.

1. **Observe** $\text{Binom}(n, 2/3)$ elements of N , which we denote by $A \subseteq N$.

Determine maximum weight independent set $\text{OPT}_A = \{f_{i_1}, \dots, f_{i_p}\}$ in A where $1 \leq i_1 < \dots < i_p \leq n$. Define $P_j = \{f_k \mid k \in \{i_{j-1}, \dots, i_j\}\} \setminus A$ for $j \in \{1, \dots, p+1\}$, where we set $i_0 = 0, i_{p+1} = n$. Let

$$\begin{aligned} \mathcal{P}_{\text{odd}}(A) &= \{P_j \mid j \in \{1, \dots, p+1\}, j \text{ odd}\}, \\ \mathcal{P}_{\text{even}}(A) &= \{P_j \mid j \in \{1, \dots, p+1\}, j \text{ even}\}. \end{aligned}$$

If $\text{OPT}_A = \emptyset$ **then** set $\mathcal{P} = \{N \setminus A\}$,

else set $\mathcal{P} = \mathcal{P}_{\text{odd}}(A)$ with probability 0.5, otherwise set $\mathcal{P} = \mathcal{P}_{\text{even}}(A)$.

2. **Apply** to each set $P \in \mathcal{P}$ an e -approximate classical secretary algorithm to obtain an element $g_P \in P$.
Return $\{g_P \mid P \in \mathcal{P}\}$.
-

Notice that applying a standard secretary algorithm to the sets of \mathcal{P} in step 2 can easily be performed by running $|\mathcal{P}|$ many e -approximate secretary algorithms in parallel, one for each set $P \in \mathcal{P}$. Elements are drawn one by one in the second phase, and they are forwarded to the secretary algorithm corresponding to the set P that contains the drawn element, and are discarded if no set of \mathcal{P} contains the element. Furthermore, observe that A contains each element of N independently with probability $2/3$.

We start by observing that Algorithm 2 returns an independent set.

Lemma 5. *Let $A \subseteq N$ with $\text{OPT}_A \neq \emptyset$ and let $\mathcal{P} \in \{\mathcal{P}_{\text{even}}(A), \mathcal{P}_{\text{odd}}(A)\}$. For each $P \in \mathcal{P}$, let g_P be any element in P . Then $\{g_P \mid P \in \mathcal{P}\} \in \mathcal{I}$.*

Proof. Let $I = \{g_P \mid P \in \mathcal{P}\}$ be a set as stated in the lemma. Notice that for any two elements $f_k, f_\ell \in I$ with $k < \ell$ we have $|\text{OPT}_A \cap \{f_k, f_{k+1}, \dots, f_\ell\}| \geq 2$. Now consider a set $L \in \mathcal{L}$ corresponding to one of the constraints of the underlying laminar matroid. By the above observation and since L is consecutively numbered, at least one of the following holds: (i) $|L \cap I| = 1$, or (ii) $|L \cap \text{OPT}_A| \geq |L \cap I|$. If case (i) holds, then the constraint corresponding to L is not violated since we assumed $b_L \geq 1$. If (ii) holds, then L is also not violated since $|L \cap I| \leq |L \cap \text{OPT}_A| \leq b_L$ because $\text{OPT}_A \in \mathcal{I}$. Hence $I \in \mathcal{I}$. \square

The following lemma presents a concise reasoning allowing us to show that Algorithm 2 is a constant-factor approximation. When presenting our $3\sqrt{3}e$ -approximation, we show a strengthening of this analysis.

Theorem 6. *Algorithm 2 is a $27e/2$ -approximation for the laminar matroid secretary problem.*

Proof. Let $\text{OPT} \in \mathcal{I}$ be the maximum weight independent set in N , i.e., the offline optimum. Furthermore, let I be the set returned by Algorithm 2, and let $f \in \text{OPT}$. We say that f is *solitary* if there is a set $P \in \mathcal{P}$ with $P \cap \text{OPT} = \{f\}$. Similarly we call $P \in \mathcal{P}$ *solitary* if $|P \cap \text{OPT}| = 1$. We prove the theorem by showing that each element $f \in \text{OPT}$ is solitary with probability $\geq 2/27$. This indeed implies the theorem since we can do the following type of accounting. Let X_f be the random variable which is zero if f is not solitary, and otherwise if f is solitary, X_f is equal to the weight of the element $g \in I$ that was chosen by the algorithm out of the set P that contains f . By only considering the weights of elements chosen in solitary sets \mathcal{P} we obtain

$$\mathbf{E}[w(I)] \geq \sum_{f \in \text{OPT}} \mathbf{E}[X_f]. \quad (4)$$

However, if each element $f \in \text{OPT}$ is solitary with probability $2/27$, we obtain $\mathbf{E}[X_f] \geq \frac{2w(f)}{27e}$, because the classical secretary algorithm will choose with probability $1/e$ the maximum weight element of the set P that contains the solitary element f . Combining this with (4) yields $\mathbf{E}[w(I)] \geq \frac{2}{27e} w(\text{OPT})$ as desired. It remains to show that each $f \in \text{OPT}$ is solitary with probability $\geq 2/27$.

Let $f_i \in \text{OPT}$. We assume that OPT contains an element with a lower index than i and one with a higher index than i . The cases of f_i being the element with highest or lowest index in OPT follow analogously. We denote by $f_j \in \text{OPT}$ the element of OPT with the largest index $j < i$. Similarly, let $f_k \in \text{OPT}$ be the element of OPT with the smallest index $k > i$. One well-known matroidal property that we use is $\text{OPT} \cap A \subseteq \text{OPT}_A$. Hence, if $f_j, f_k \in A$ then also $f_j, f_k \in \text{OPT}_A$, and if furthermore $f_i \notin A$, then f_i will be the only element of OPT in the set $P \in \mathcal{P}_{\text{odd}}(A) \cup \mathcal{P}_{\text{even}}(A)$ that contains f_i . Hence, if the coin flip in Algorithm 2 chooses the family $\mathcal{P} \in \{\mathcal{P}_{\text{odd}}(A), \mathcal{P}_{\text{even}}(A)\}$ that contains P , then f_i is solitary. To summarize, f_i is solitary if $f_j, f_k \in A$, $f_i \notin A$ and the coin flip for \mathcal{P} turns out right. This happens with probability $(\frac{2}{3})^2 \cdot (1 - \frac{2}{3}) \cdot \frac{1}{2} = \frac{2}{27}$. □

In the following we discuss how Algorithm 2 and its analysis can be improved to obtain a $3\sqrt{3}e$ -approximation.

One conservative aspect of the analysis used in Lemma 6 is that we only account for the contribution of solitary elements. Additionally, a drawback of Algorithm 2 itself is that about half of the elements of $N \setminus A$ are ignored since elements are selected from only one of the two families $\mathcal{P}_{\text{odd}}(A), \mathcal{P}_{\text{even}}(A)$. We start by describing a stronger way to define a partition \mathcal{P} of $N \setminus A$ and reduce the problem to a matroid secretary problem on the unitary partition matroid defined on \mathcal{P} .

For any independent set $I \in \mathcal{I}$, we define a partition $\tilde{\mathcal{P}}(I)$ of N as follows. If $I = \emptyset$, we set $\tilde{\mathcal{P}}(I) = \{N\}$. Otherwise $\tilde{\mathcal{P}}(I)$ contains a set $N_f \subseteq N$ for each element $f \in I$, i.e., $\tilde{\mathcal{P}}(I) = \{N_f \mid f \in I\}$. To define the partition $\tilde{\mathcal{P}}(I)$, we specify to which set N_f an element $f_i \in N$ belongs. Let $L \in \mathcal{L}$ be the smallest set that contains f_i and such that $L \cap I \neq \emptyset$. Such a set must exist since $N \in \mathcal{L}$ by assumption. If $L \cap I$ contains at least one element f_j with $j \leq i$, then let j be the largest index such that $j \leq i$ and $f_j \in L \cap I$. Otherwise let j be the smallest index satisfying $j > i$ and $f_j \in L \cap I$. We assign the element f_i to N_{f_j} .

Notice that in any case, j is either the largest index $j \leq i$ with $f_j \in I$ or the smallest index $j > i$ with $f_j \in I$. Again, we are interested to define a partition only on elements $N \setminus A$ not drawn in the first phase. We therefore define for any $A \subseteq N$ the partition $\mathcal{P}(A) = \{\tilde{P} \setminus A \mid \tilde{P} \in \tilde{\mathcal{P}}(\text{OPT}_A)\}$. Algorithm 3 describes our $3\sqrt{3}e$ -approximation.

We first show that the set returned by Algorithm 3 is indeed independent. For this, we start by observing a basic property of the sets N_f forming the underlying partition $\tilde{\mathcal{P}}(\text{OPT}_A) = \{N_f \mid f \in \text{OPT}_A\}$.

Lemma 7. *Let $I \in \mathcal{I}$ with $I \neq \emptyset$. Each set N_{f_i} of the partition $\tilde{\mathcal{P}}(I) = \{N_{f_i} \mid f_i \in I\}$ is of the form $N_{f_i} = \{f_j, f_{j+1}, \dots, f_k\}$ for some $1 \leq j \leq i \leq k \leq n$.*

Algorithm 3 A $3\sqrt{3}e$ -approximation for laminar matroids.

1. **Observe** $\text{Binom}(n, 1/\sqrt{3})$ elements of N , which we denote by $A \subseteq N$.
Determine maximum weight independent set OPT_A in A .
 2. **Apply** to each set $P \in \mathcal{P}(A)$ an e -approximate classical secretary algorithm to obtain $g_P \in P$.
Return $\{g_P \mid P \in \mathcal{P}(A)\}$.
-

Proof. By definition of N_{f_i} , we clearly have $f_i \in N_{f_i}$. Hence, all that remains to be shown is that whenever $f_p \in N_{f_i}$, then $f_q \in N_{f_i}$ for any q between i and p , i.e., either $i < q < p$ or $p < q < i$. In the following we distinguish these two cases. For any element $f \in N$, we denote by $L_f \in \mathcal{L}$ the smallest set $L \in \mathcal{L}$ that contains f and satisfies $I \cap L \neq \emptyset$.

Case $p < q < i$. Since $f_p \in N_{f_i}$, there is no element $f_\ell \in L_{f_p} \cap I$ with $\ell < i$. Furthermore, $f_p, f_i \in L_{f_p}$ implies $f_q \in L_{f_p}$, because L_{f_p} contains a sequence of consecutively numbered elements. As a consequence, there is also no element $f_\ell \in L_{f_q} \cap I$ with $\ell < i$, because $L_{f_q} \subseteq L_{f_p}$ due to laminarity and the fact that L_{f_q} is the smallest set in \mathcal{L} containing f_q and satisfying $I \cap L_{f_q} \neq \emptyset$. Hence $f_q \in N_{f_i}$.

Case $i < q < p$. As in the previous case we have $f_i \in L_{f_q} \subseteq L_{f_p}$, and there is no ℓ with $i < \ell < p$ such that $f_\ell \in I$, using again $f_p \in N_{f_i}$. Thus, $f_q \in N_{f_i}$. \square

The next lemma implies that Algorithm 3 returns an independent set.

Lemma 8. *Let $I \subseteq \mathcal{I}$ and let $J \subseteq N$ with $|J \cap \tilde{P}| \leq 1 \forall \tilde{P} \in \tilde{\mathcal{P}}(I)$. Then $J \in \mathcal{I}$.*

Proof. To show $J \in \mathcal{I}$ we fix any $L \in \mathcal{L}$ and show that J satisfies the constraint imposed on L , i.e., $|J \cap L| \leq b_L$. If $I \cap L = \emptyset$, then all elements in L belong to the same set of the partition $\tilde{\mathcal{P}}(I)$. Hence $|J \cap L| \leq 1$, and the constraint corresponding to L is not violated since by assumption $b_L \geq 1$. Hence, assume $I \cap L \neq \emptyset$. Notice that in this case every element in L will be assigned to a set N_f for $f \in I \cap L$, i.e.,

$$L \subseteq \bigcup_{f \in I \cap L} N_f. \quad (5)$$

Since at most one element is chosen out of each N_f we have

$$|J \cap L| \leq |I \cap L| \leq b_L,$$

where the second inequality follows from $I \in \mathcal{I}$. \square

Since the family $\mathcal{P}(A)$ consists of subsets of the partition $\tilde{\mathcal{P}}(\text{OPT}_A)$, the above lemma implies:

Corollary 9. *Algorithm 3 returns an independent set.*

It remains to show the claimed approximation guarantee.

Theorem 10. *Algorithm 3 is a $3\sqrt{3}e$ -approximation for the laminar matroid secretary problem.*

Proof. Let $\text{OPT}_{\mathcal{P}(A)}$ be the optimum solution of the matroid secretary problem on $N \setminus A$ constrained by the partition matroid $\mathcal{P}(A)$. Let I be the solution returned by Algorithm 3. Since Algorithm 3 applies an e -approximate secretary algorithm to each set of $\mathcal{P}(A)$, we have

$$\mathbf{E}[w(I)] \geq \frac{1}{e} \cdot \mathbf{E}[w(\text{OPT}_{\mathcal{P}(A)})]. \quad (6)$$

For $f \in N \setminus A$, we denote by P_f the set in the family $\mathcal{P}(A)$ that contains f . We have,

$$\begin{aligned}
\mathbf{E}[w(\text{OPT}_{\mathcal{P}(A)})] &= \mathbf{E} \left[\sum_{P \in \mathcal{P}(A)} \max_{f \in P} w(f) \right] \geq \mathbf{E} \left[\sum_{\substack{P \in \mathcal{P}(A), \\ |P \cap \text{OPT}| \geq 1}} \max_{f \in P} w(f) \right] \\
&\geq \mathbf{E} \left[\sum_{\substack{P \in \mathcal{P}(A), \\ |P \cap \text{OPT}| \geq 1}} \sum_{f \in P \cap \text{OPT}} \frac{w(f)}{|P \cap \text{OPT}|} \right] \\
&= \mathbf{E} \left[\sum_{f \in \text{OPT} \setminus A} \frac{w(f)}{|P_f \cap \text{OPT}|} \right]. \tag{7}
\end{aligned}$$

Similar to the proof of Theorem 6 we use an accounting based on the elements of the offline optimum OPT. For each $f \in \text{OPT}$ we define a random variable $Z(f)$ as follows:

$$Z(f) = \begin{cases} 0 & \text{if } f \in A, \\ \frac{1}{|P_f \cap \text{OPT}|} & \text{otherwise.} \end{cases}$$

Together with (6) and (7) we thus obtain

$$\mathbf{E}[w(I)] \geq \frac{1}{e} \sum_{f \in \text{OPT}} w(f) \mathbf{E}[Z(f)].$$

Hence, to show that Algorithm 3 is a $3\sqrt{3}e$ -approximation, it suffices to show

$$\mathbf{E}[Z(f)] \geq \frac{1}{3\sqrt{3}} \quad \forall f \in \text{OPT}. \tag{8}$$

For proving (8), we want to be able to treat all elements $f_i \in \text{OPT}$ the same way, independently of the index i . In particular, we want to avoid special treatments for indices i that are close to the border, i.e., either close to 1 or n . Therefore we make the following assumptions, which do not change the way in which the algorithm behaves: assume that there are infinitely many dummy coloop⁶ elements (with zero weight) denoted as $C = \{\dots, f_{-2}, f_{-1}, f_0\} \cup \{f_{n+1}, f_{n+2}, \dots\}$. The new (infinite) laminar matroid M' is associated to the laminar family $\mathcal{L}' = \mathcal{L} \cup \{N \cup C\}$, where $N \cup C$ has no bound on the cardinality.

The optimum OPT' of M' equals C union the optimum $\text{OPT} = \{f_{i_1}, \dots, f_{i_p}\}$ of the original matroid. If we run the algorithm on this modified infinite matroid—assuming that every element, original or dummy, belongs to A with probability $1/\sqrt{3}$ —and then remove the dummy elements from its output, we recover the output that we would have obtained had we used the real matroid.

We fix an element $f_{i_r} \in \text{OPT}$ and prove (8) for this element in the following. To have f_{i_j} defined for every integer j , even outside of $\{1, \dots, p\}$, we set $i_j = j$ for $j \leq 0$, and $i_j = n - p + j$ for $j > p$. Hence, $\text{OPT}' = \{f_{i_j} \mid j \text{ integer}\}$. Furthermore, to simplify the exposition and to explain later why $1/\sqrt{3}$ was chosen to be the probability of including elements in A , we denote by q the probability that an element is contained in A .

For every pair of natural numbers $s, t \geq 0$, define $\mathcal{E}_{s,t}$ as the event that the following occurs simultaneously:

⁶A coloop is an element that is in every base of the matroid, or in other words, a coloop element can be added to any independent set without destroying independence.

- (i) $f_{i_r} \notin A$,
- (ii) $f_{i_{r-1-s}}$ is the last element of OPT' before f_{i_r} that is in A , and
- (iii) $f_{i_{r+1+t}}$ is the first element in OPT' after f_{i_r} that is in A .

In other word, $\mathcal{E}_{s,t}$ is the event that $f_{i_{r-(s+1)}} \in A$; $f_{i_{r-s}}, \dots, f_{i_{r+t}} \notin A$; and $f_{i_{r+(t+1)}} \in A$.

From this point on we condition on the event $\mathcal{E}_{s,t}$. Consider $\mathcal{L}'_{f_{i_r}} = \{L \in \mathcal{L}' \mid f_{i_r} \in L\}$. Since \mathcal{L}' is a laminar family, $\mathcal{L}'_{f_{i_r}}$ is a chain. Let $L \in \mathcal{L}'_{f_i}$ be the smallest set in \mathcal{L}'_{f_i} with $(L \cap \text{OPT}') \setminus A \neq \emptyset$; or equivalently, $\{f_{i_{r-(s+1)}}, f_{i_{r+(t+1)}}\} \cap L \neq \emptyset$. We claim that

$$\mathbf{E}[Z(f_{i_r}) \mid \mathcal{E}_{s,t}] \geq q \sum_{k=0}^{\infty} \frac{1}{s+t+1+k} (1-q)^k. \quad (9)$$

To prove (9), we distinguish two cases: (a) $f_{i_{r-(s+1)}} \notin L$ and (b) $f_{i_{r-(s+1)}} \in L$.

In the first case, let $K \geq 0$ be the random variable counting the number of consecutive elements in $(f_{i_j})_j$ immediately after $f_{i_{r+t+1}}$ that are not contained in A . In other words, $f_{i_{r+(t+1)+K+1}}$ is the first element of OPT' after $f_{i_{r+(t+1)}}$ that is in A . Note that conditioned on $\mathcal{E}_{s,t}$ and on the variable K , the set $P \in \mathcal{P}(A)$ to which f_{i_r} belongs must be a subset of $Q = \{f_{i_{r-(s+1)+1}}, \dots, f_{i_{r+t+K+2-1}}\}$.

In particular, $Q \cap \text{OPT}' \subseteq \{f_{i_{r-s}}, \dots, f_{i_{r+t+K+1}}\}$. Recalling that $f_{i_{r+t+1}} \in A$ and $P(f_{i_r}) \subseteq N \setminus A$, we conclude $|P(f_{i_r}) \cap \text{OPT}'| \leq |Q \cap \text{OPT}'| - 1$, and hence

$$Z(f_{i_r}) = \frac{1}{|P(f_{i_r}) \cap \text{OPT}'|} \geq \frac{1}{|Q \cap \text{OPT}'| - 1} = \frac{1}{t+s+1+K}.$$

Therefore,

$$\mathbf{E}[Z(f_{i_r}) \mid \mathcal{E}_{s,t}] \geq \sum_{k=0}^{\infty} \mathbf{E}[Z(f_{i_r}) \mid \mathcal{E}_{s,t}, K=k] \cdot \Pr(K=k) \geq \sum_{k=0}^{\infty} \frac{1}{t+s+1+k} q(1-q)^k,$$

which proves (9) for case (a). The proof of the claim in case (b) is analogous, but in that case we define $K \geq 0$ as the random variable counting the number of consecutive elements in $(f_{i_j})_j$ immediately before $f_{i_{r-(s+1)}}$ that are outside A .

Based on (9), we can conclude the proof of the theorem as follows. Since all events $(\mathcal{E}_{s,t})_{s,t \geq 0}$ are disjoint and $\Pr(\mathcal{E}_{s,t}) = q^2(1-q)^{s+t+1}$, we have

$$\begin{aligned} \mathbf{E}[Z(f_{i_r})] &= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \mathbf{E}[Z(f_{i_r}) \mid \mathcal{E}_{s,t}] \Pr(\mathcal{E}_{s,t}) \geq q^3 \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{s+t+k+1} (1-q)^{s+t+k+1} \\ &= q^3 \sum_{\ell=0}^{\infty} \binom{\ell+2}{2} \frac{1}{\ell+1} (1-q)^{\ell+1} = \frac{q^3}{2} \sum_{\ell=0}^{\infty} (\ell+2) (1-q)^{\ell+1} \\ &\stackrel{(\star)}{=} \frac{q^3}{2} \frac{(1-q)(1+q)}{q^2} = \frac{1}{2} q(1-q^2), \end{aligned} \quad (10)$$

where equality (\star) is obtained by setting $x = 1 - q$ in

$$\sum_{\ell=0}^{\infty} (\ell+2) x^{\ell+1} = \frac{d}{dx} \left(\sum_{\ell=0}^{\infty} x^{\ell+2} \right) = \frac{d}{dx} \frac{x^2}{1-x} = \frac{x(2-x)}{(1-x)^2}.$$

Finally, $q = 1/\sqrt{3}$ is chosen to maximize $q(1-q^2)/2$ among all values in $[0, 1]$, and implies by (10),

$$\mathbf{E}[Z(f_{i_r})] \geq \frac{1}{3\sqrt{3}},$$

thus proving (8). □

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